

WEIGHT-MONODROMY CONJECTURE OVER EQUAL CHARACTERISTIC LOCAL FIELDS

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ABSTRACT. The aim of this paper is to study certain properties of the weight spectral sequences of Rapoport-Zink by a specialization argument. By reducing to the case over finite fields previously treated by Deligne, we prove that the weight filtration and the monodromy filtration defined on the l -adic étale cohomology coincide, up to shift, for proper smooth varieties over equal characteristic local fields. We also prove that the weight spectral sequences degenerate at E_2 in any characteristic without using log geometry. Moreover, as an application, we give a modulo $p > 0$ reduction proof of a Hodge analogue previously considered by Steenbrink.

1. INTRODUCTION

The aim of this paper is to study certain properties of the weight spectral sequences of Rapoport-Zink by a specialization argument. Let K be a discrete valuation field with ring of integers \mathcal{O}_K and residue field F . Assume that \mathcal{O}_K is henselian. Let π be a uniformizer of K , and l a prime number different from the characteristic of F . Let \mathfrak{X} be a regular scheme proper and flat over \mathcal{O}_K such that the generic fiber $\mathfrak{X}_K := \mathfrak{X} \otimes_{\mathcal{O}_K} K$ is a proper smooth variety of dimension n over K , and the special fiber $\mathfrak{X}_F := \mathfrak{X} \otimes_{\mathcal{O}_K} F$ is a divisor of \mathfrak{X} with simple normal crossings. Such \mathfrak{X} is called a *proper strictly semistable scheme* over \mathcal{O}_K . Let $\mathfrak{X}^{(k)}$ be the disjoint union of k by k intersections of the irreducible components of \mathfrak{X}_F .

The *weight spectral sequence* of Rapoport-Zink is the following $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ -equivariant spectral sequence relating the l -adic étale cohomology of the special fiber and the generic fiber ([RZ], Satz 2.10):

$$E_1^{-r, w+r} = \bigoplus_{k \geq \max\{0, -r\}} H_{\mathrm{\acute{e}t}}^{w-r-2k}(\mathfrak{X}_{F^{\mathrm{sep}}}^{(2k+r+1)}, \mathbb{Q}_l(-r-k)) \Rightarrow H_{\mathrm{\acute{e}t}}^w(\mathfrak{X}_{K^{\mathrm{sep}}}, \mathbb{Q}_l),$$

where $\mathfrak{X}_{F^{\mathrm{sep}}}^{(k)} := \mathfrak{X}^{(k)} \otimes_F F^{\mathrm{sep}}$, $\mathfrak{X}_{K^{\mathrm{sep}}} := \mathfrak{X}_K \otimes_K K^{\mathrm{sep}}$, and F^{sep} (resp. K^{sep}) denotes the separable closure of F (resp. K). We have a natural map, called the *monodromy operator* $N: E_1^{i,j}(1) \rightarrow E_1^{i+2,j-2}$ such that $N^r: E_1^{-r, w+r}(r) \xrightarrow{\cong} E_1^{r, w-r}$ is an isomorphism for all r, w . Note that, the monodromy operator N is a combination of identity maps and zero maps, and the differentials $d_1^{i,j}$ on the E_1 -terms

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are explicitly described in terms of restriction morphisms and Gysin morphisms (for details, see [RZ], [Il1], [Il2], [SaT]).

The main theorem of this paper is as follows.

Theorem 1.1. *Let notation be as above.*

- (1) ([Na], Theorem 0.1) *The weight spectral sequence degenerates at E_2 .*
- (2) *If K is of equal characteristic, the monodromy operator N induces an isomorphism $N^r: E_2^{-r, w+r}(r) \xrightarrow{\cong} E_2^{r, w-r}$ for all r, w .*

When F is a finite field, the first part of Theorem 1.1 is an easy consequence of the Weil conjectures ([RZ], Satz 2.10, see also Proposition 2.1). The general case was already proved by C. Nakayama by specializing log schemes ([Na], Theorem 0.1). He essentially used his construction of the weight spectral sequences for log schemes. In this paper, we give another proof without using log geometry.

The second part of Theorem 1.1 is sometimes called the *weight-monodromy conjecture*. Namely, we have the following conjecture in any characteristic ([De1], [De3], see also [RZ], [Il1], [Il2]).

Conjecture 1.2 (Weight-monodromy conjecture). *Let notation be as above. Then, the monodromy operator N induces an isomorphism $N^r: E_2^{-r, w+r}(r) \xrightarrow{\cong} E_2^{r, w-r}$ for all r, w .*

According to Theorem 1.1, Conjecture 1.2 holds if K is of equal characteristic. In mixed characteristic, Conjecture 1.2 is known to hold if either \mathfrak{X}_K is an abelian variety or $\dim \mathfrak{X}_K \leq 2$ ([SGA7-I], [RZ], [dJ], [SaT]). Although some partial results were obtained ([It2], [It3]), Conjecture 1.2 is still open in general (see also Remark 2.4).

It is well-known that, if the residue field F is finite, Conjecture 1.2 is equivalent to another conjecture (Conjecture 2.3) on the monodromy filtration and the weight filtration defined on the l -adic étale cohomology of the geometric generic fiber $\mathfrak{X}_{K^{\text{sep}}}$ (Proposition 2.5, Remark 2.6). When $\mathfrak{X}/\mathcal{O}_K$ is the henselization of a family of varieties over a curve over a finite field, Deligne proved Conjecture 2.3, and hence he essentially proved Conjecture 1.2 although the weight spectral sequence was not constructed at that time (Remark 2.4, [De3], Théorème 1.8.4).

In this paper, we give a proof of Theorem 1.1 by a specialization argument. The key geometric construction is given in §4 using Néron's blowing up. Then, the first part of Theorem 1.1 easily follows in all characteristics (Proposition 5.1). For the second part of Theorem 1.1, we reduce the general case to the case treated by Deligne in [De3]. This argument is valid only in the equal characteristic case (Remark 6.2). Note that the possibility of such an argument was pointed out by Illusie in [Il2], 8.7, and a similar specialization argument for Conjecture 2.3 was considered by Terasoma when F is a finite field ([Te]). In §7, we also give

some consequences. In §7.1, we prove the equal characteristic case of Conjecture 2.3 by a similar specialization argument (Proposition 7.1). Moreover, in §7.2, we give a modulo $p > 0$ reduction proof of a Hodge analogue previously considered by Steenbrink (Proposition 7.3).

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2. MONODROMY FILTRATION AND WEIGHT FILTRATION

In this section, we recall the monodromy filtration and the weight filtration defined on the l -adic étale cohomology of proper smooth varieties over local fields. We also recall a conjecture on these two filtrations (Conjecture 2.3) which is equivalent to Conjecture 1.2 in the semistable reduction case.

As in §1, let K be a discrete valuation field with ring of integers \mathcal{O}_K and residue field F . Assume that \mathcal{O}_K is henselian. Let π be a uniformizer of K , and l a prime number different from the characteristic of F . Let X_K be a proper smooth variety of dimension n over K , $V := H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ the l -adic étale cohomology of $X_{K^{\text{sep}}} := X_K \otimes_K K^{\text{sep}}$, and $\rho: \text{Gal}(K^{\text{sep}}/K) \rightarrow \text{GL}(V)$ the action of $\text{Gal}(K^{\text{sep}}/K)$ on V .

2.1. Monodromy filtration. First of all, recall that we have the exact sequence $1 \rightarrow I_K \rightarrow \text{Gal}(K^{\text{sep}}/K) \rightarrow \text{Gal}(F^{\text{sep}}/F) \rightarrow 1$, where I_K is called the *inertia group* of K . The pro- l -part of I_K is isomorphic to $\mathbb{Z}_l(1)$ by a natural map

$$t_l: I_K \ni \sigma \mapsto \left(\frac{\sigma(\pi^{1/l^m})}{\pi^{1/l^m}} \right)_m \in \varprojlim \mu_{l^m} =: \mathbb{Z}_l(1),$$

where μ_{l^m} is the group of l^m -th roots of unity in K^{sep} ([Se]). By Grothendieck's monodromy theorem ([ST], Appendix, see also [SGA7-I], I, Variante 1.3), there is an open subgroup $J \subset I_K$ and a nilpotent map $N: V(1) \rightarrow V$ such that $\rho(\sigma) = \exp(t_l(\sigma)N)$ for all $\sigma \in J \subset I_K$. Here N is a nilpotent map means that there is $r \geq 1$ such that $N^r: V(r) \rightarrow V$ is a zero map, where $V(r)$ denotes the r -th Tate twist of V . The map N is called the *monodromy operator* on V .

The *monodromy filtration* M on V is a unique increasing filtration such that $M_{-k}V = 0$, $M_kV = V$ for sufficiently large k , $N(M_kV(1)) \subset M_{k-2}V$ for all k ,

and N induces an isomorphism $N^k: \mathrm{Gr}_k^M V(k) \xrightarrow{\cong} \mathrm{Gr}_{-k}^M V$ for all $k \geq 0$, where $\mathrm{Gr}_k^M V := M_k V / M_{k-1} V$ ([De3], I, (1.7.2)).

2.2. Weight filtration. Usually, the notion of weights is considered when F is a finite field. Here we consider it in a slightly general situation. Assume that F is a purely inseparable extension of a finitely generated extension of a prime field (i.e. \mathbb{F}_p or \mathbb{Q}). It is always possible to find a finitely generated \mathbb{Z} -algebra A contained in F such that F is a purely inseparable extension of the field of fractions $\mathrm{Frac} A$ of A .

We say a continuous l -adic representation ρ of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ has *weight* k if ρ comes from a smooth l -adic sheaf \mathcal{F} on an open dense set $U \subset \mathrm{Spec} A$ by base change and \mathcal{F} has weight k in the usual sense (for example, see [De1], [De3], (1.2)). Namely, for all closed points $s \in U$, the eigenvalues of the geometric Frobenius element at s acting on $\mathcal{F}_{\bar{s}}$ are algebraic integers such that the complex absolute values of their complex conjugates are $|\kappa(s)|^{k/2}$, where $|\kappa(s)|$ denotes the order of the residue field $\kappa(s)$ at s . It is easy to see that this definition is independent of the choice of A , U , \mathcal{F} . By the Weil conjectures ([De2], [De3]), for a proper smooth variety Z over F , the l -adic étale cohomology $H_{\mathrm{ét}}^k(Z_{F^{\mathrm{sep}}}, \mathbb{Q}_l)$ has weight k in this sense.

Let us go back to the situation in the beginning of this section. As above, assume that F is a purely inseparable extension of a finitely generated extension of a prime field. The *weight filtration* W on V is a unique increasing filtration such that $W_{-k} V = 0$, $W_k V = V$ for sufficiently large k , the action of I_K on $\mathrm{Gr}_k^W V := W_k V / W_{k-1} V$ factors through a finite quotient, and, after replacing K by a finite extension of it, $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $\mathrm{Gr}_k^W V$ and this action has weight k ([De1], [De3], I, (1.7.5), for the existence of the weight filtration, see §2.3 below).

2.3. An application to the weight spectral sequences. Let notation be as in the beginning of this section. Assume that the residue field F is a purely inseparable extension of a finitely generated extension of a prime field, and there exists a proper strictly semistable scheme \mathfrak{X} over \mathcal{O}_K such that the generic fiber \mathfrak{X}_K is isomorphic to X_K . Then, each $E_1^{-r, w+r}$ of the weight spectral sequence has weight $(w - r - 2k) - 2(-r - k) = w + r$ by the Weil conjectures (§2.2). Therefore, the filtration on $V := H_{\mathrm{ét}}^w(X_{K^{\mathrm{sep}}}, \mathbb{Q}_l)$ induced by the weight spectral sequence is nothing but the weight filtration in §2.2. This proves the existence of the weight filtration on V in the semistable reduction case. In general, we may use de Jong's alteration to reduce to the semistable reduction case (Remark 7.2, see also the Introduction of [dJ]. See [GH] for the construction of the weight spectral sequence via the weight filtration on nearby cycles).

Here we give an immediate application of the notion of weights to the weight spectral sequences.

Proposition 2.1. *Let notation be as in §1. Assume that the residue field F is a purely inseparable extension of a finitely generated extension of a prime field. Then, the weight spectral sequence degenerates at E_2 .*

Proof. Since a map between l -adic $\text{Gal}(F^{\text{sep}}/F)$ -representations with different weights is zero, we have $d_r^{i,j} = 0$ for $r \geq 2$. Hence the weight spectral sequence degenerates at E_2 . \square

Remark 2.2. Proposition 2.1 is a special case of the first assertion of Theorem 1.1 (see also [RZ], Satz 2.10, [Na], Theorem 0.1). In §5, we will prove the general case without any assumption on residue fields by a specialization argument.

2.4. A conjecture. The following conjecture is also called the weight-monodromy conjecture. Sometimes, this is called *Deligne's conjecture on the purity of monodromy filtration* in the literature.

Conjecture 2.3 ([De1], [De3], see also [RZ], [Ra], [Il1], [Il2]). *Let notation be as in the beginning of this section. Recall that X_K is a proper smooth variety of dimension n over K and $V := H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l)$. Assume that the residue field F is a purely inseparable extension of a finitely generated extension of a prime field. Let M be the monodromy filtration on V (§2.1), and W the weight filtration on V (§2.2). Then, we have $M_k V = W_{w+k} V$ for all k .*

Remark 2.4. Conjecture 2.3 is known to hold if either X_K is an abelian variety or $\dim X_K \leq 2$ ([SGA7-I], [RZ], [dJ], [SaT]). In mixed characteristic and in dimension ≥ 3 , Conjecture 2.3 is open in general. The author proved Conjecture 2.3 for certain threefolds ([It2]) and p -adically uniformized varieties ([It3]). In characteristic $p > 0$, if X_K/K is the henselization of a family of varieties over a curve over a finite field, Deligne proved Conjecture 2.3 in his proof of the Weil conjectures ([De3], Théorème 1.8.4). Note that, in §7.1, we will prove the equal characteristic case of Conjecture 2.3 by reducing to the case treated by Deligne.

Note that we can state Conjecture 2.3 for a proper smooth variety X_K over K with an assumption on the residue field F . On the other hand, we can state Conjecture 1.2 for a proper strictly semistable scheme \mathfrak{X} over \mathcal{O}_K without any assumption on F . The relation between these two conjectures is as follows (see also Remark 2.6 below).

Proposition 2.5. *Let notation be as in the beginning of this section. Assume that the residue field F is a purely inseparable extension of a finitely generated extension of a prime field, and there exists a proper strictly semistable scheme \mathfrak{X} over \mathcal{O}_K such that the generic fiber \mathfrak{X}_K of \mathfrak{X} is isomorphic to X_K . Then, Conjecture 2.3 for X_K and Conjecture 1.2 for \mathfrak{X} are equivalent to each other.*

Proof. As we saw in §2.3, the weight spectral sequence degenerates at E_2 (Proposition 2.1), and it induces the weight filtration W on V by the Weil conjectures. The monodromy operator N in §2.1 is induced from the monodromy operator

N on the weight spectral sequence (§1). Therefore, according to the definition of the monodromy filtration (§2.1), Conjecture 2.3 holds for X_K if and only if $N^r: E_2^{-r, w+r}(r) \rightarrow E_2^{r, w-r}$ is an isomorphism for all r, w . \square

Remark 2.6. If we vary K , Conjecture 1.2 and Conjecture 2.3 are equivalent to each other. Precisely speaking, by using the geometric construction in §4, we can prove that Conjecture 2.3 implies Conjecture 1.2 (see Remark 6.2). Conversely, by de Jong's alteration, we can also prove that Conjecture 1.2 implies Conjecture 2.3 (see Remark 7.2).

3. SOME LEMMAS

Here we list some lemmas which immediately follow from the basic properties of the weight spectral sequences. Let notation be as in §1. Recall that \mathfrak{X} is a proper strictly semistable scheme over a henselian discrete valuation ring \mathcal{O}_K . In this section, we do not put any assumption on the residue field F .

Lemma 3.1. *The weight spectral sequence $E_1^{i,j} \Rightarrow E^{i+j}$ of $\mathfrak{X}/\mathcal{O}_K$ degenerates at E_2 if and only if the following equality holds:*

$$\sum_{i,j} \dim_{\mathbb{Q}_l} E_2^{i,j} = \sum_k \dim_{\mathbb{Q}_l} E^k.$$

Proof. It is easy to see that the above equality holds if and only if all maps $d_r^{i,j}$ for $r \geq 2$ are zero. Hence we have the assertion. \square

Lemma 3.2. *Conjecture 1.2 depends only on the geometric special fiber of \mathfrak{X} in the following sense. Namely, if $\mathfrak{X}/\mathcal{O}_K$ (resp. $\mathfrak{X}'/\mathcal{O}_{K'}$) is a proper strictly semistable scheme over a henselian discrete valuation ring \mathcal{O}_K (resp. $\mathcal{O}_{K'}$) such that there exists a separably closed field F containing the residue fields of \mathcal{O}_K and $\mathcal{O}_{K'}$, and $\mathfrak{X} \otimes_{\mathcal{O}_K} F \cong \mathfrak{X}' \otimes_{\mathcal{O}_{K'}} F$, then Conjecture 1.2 for $\mathfrak{X}/\mathcal{O}_K$ and $\mathfrak{X}'/\mathcal{O}_{K'}$ are equivalent to each other.*

Proof. In general, the E_1 -terms of the weight spectral sequence and the maps $d_1^{i,j}$, N on them are explicitly written in terms of the geometric special fiber (for details, see [RZ], Satz 2.10). Hence, the E_1 -terms and the maps $d_1^{i,j}$, N on them are the same for $\mathfrak{X}/\mathcal{O}_K$, $\mathfrak{X}'/\mathcal{O}_{K'}$. The maps in question $N^r: E_2^{-r, w+r}(r) \rightarrow E_2^{r, w-r}$ are also the same for $\mathfrak{X}/\mathcal{O}_K$, $\mathfrak{X}'/\mathcal{O}_{K'}$. Therefore, we have the assertion. \square

Lemma 3.3. *Let K' be a discrete valuation field with ring of integers $\mathcal{O}_{K'}$ such that $\mathcal{O}_{K'}$ is henselian, K' is a field extension of K , $\mathcal{O}_{K'} \cap K = \mathcal{O}_K$, and a uniformizer of K is also a uniformizer of K' . We put $\mathfrak{X}' := \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. Then the assertions of Theorem 1.1 for $\mathfrak{X}/\mathcal{O}_K$ and $\mathfrak{X}'/\mathcal{O}_{K'}$ are equivalent to each other.*

Proof. The left hand sides of the equality in Lemma 3.1 for $\mathfrak{X}/\mathcal{O}_K$, $\mathfrak{X}'/\mathcal{O}_{K'}$ are the same since $\mathfrak{X}/\mathcal{O}_K$, $\mathfrak{X}'/\mathcal{O}_{K'}$ have the same geometric special fiber. The same is true for the right hand sides since the l -adic étale cohomology of the geometric

generic fibers of $\mathfrak{X}/\mathcal{O}_K$, $\mathfrak{X}'/\mathcal{O}_{K'}$ are the same ([SGA4-III], XVI, Corollaire 1.6). Hence the assertion follows from Lemma 3.1 and Lemma 3.2. \square

Remark 3.4. For any discrete valuation field K , there is a field extension K'/K such that K' is a complete discrete valuation field with ring of integers $\mathcal{O}_{K'}$, $\mathcal{O}_{K'} \cap K = \mathcal{O}_K$, a uniformizer of K is also a uniformizer of K' , and the residue field F' of K' is perfect (see [EGA3], Chapitre 0, Proposition 10.3.1). Therefore, by Lemma 3.3, it is enough to prove Theorem 1.1 under the assumption that \mathcal{O}_K is complete and the residue field F is perfect.

Remark 3.5. We can apply Lemma 3.3 for the π -adic completion \widehat{K} of K . In this case, Lemma 3.3 shows that the assertions of Theorem 1.1 for $\mathfrak{X}/\mathcal{O}_K$ depend only on the π -adic completion $\widehat{\mathfrak{X}}$ of \mathfrak{X} . If we use the results of C. Nakayama in [Na], we can say more. We note it for the reader's convenience although we do not use it in this paper (for details, see [Na], [Il2]). Let $\mathfrak{X}/\mathcal{O}_K$ be as in §1. C. Nakayama constructed the weight spectral sequence only from the special fiber \mathfrak{X}_F with a natural log structure on it. Therefore, the assertions of Theorem 1.1 for $\mathfrak{X}/\mathcal{O}_K$ depend only on the first infinitesimal neighborhood $\mathfrak{X} \otimes_{\mathcal{O}_K} (\mathcal{O}_K/(\pi^2))$ since the log structure on \mathfrak{X}_F depends only on $\mathfrak{X} \otimes_{\mathcal{O}_K} (\mathcal{O}_K/(\pi^2))$ (e.g. [Il2], p. 309, the last paragraph).

4. A CONSTRUCTION USING NÉRON'S BLOWING UP

In this section, we give a technical construction using Néron's blowing up.

Let K be a discrete valuation field with ring of integers \mathcal{O}_K and residue field F . Let π be a uniformizer of K . Let \mathfrak{X} be a proper strictly semistable scheme of relative dimension n over \mathcal{O}_K . Then there exist a finite open covering $\{U_i\}$ of \mathfrak{X} , and, for each i , an étale morphism $f_i: U_i \rightarrow \text{Spec } \mathcal{O}_K[X_0, \dots, X_n]/(X_0 \cdots X_{r_i} - \pi)$ over \mathcal{O}_K for some r_i .

When K is of equal characteristic $p > 0$ (resp. equal characteristic 0), by Néron's blowing up ([SGA7-I], I, (0.5.2), [Ar], §4), we can write \mathcal{O}_K as a filtered inductive limit $\mathcal{O}_K = \varinjlim A_\alpha$, where each A_α is a finitely generated smooth $\mathbb{F}_p[\pi]_{(\pi)}$ -algebra (resp. $\mathbb{Q}[\pi]_{(\pi)}$ -algebra), where $\mathbb{F}_p[\pi]_{(\pi)}$ (resp. $\mathbb{Q}[\pi]_{(\pi)}$) denotes the localization of $\mathbb{F}_p[\pi]$ (resp. $\mathbb{Q}[\pi]$) at the maximal ideal generated by π . By a standard argument, for some α , there exist a scheme \mathfrak{Y} proper over $\text{Spec } A_\alpha$ with a finite open covering $\{V_i\}$, and an étale morphism $g_i: V_i \rightarrow \text{Spec } A_\alpha[X_0, \dots, X_n]/(X_0 \cdots X_{r_i} - \pi)$ over A_α for each i such that the pullback of the tuple $(\mathfrak{Y}/A_\alpha, \{V_i\}, g_i)$ by $\text{Spec } \mathcal{O}_K \rightarrow \text{Spec } A_\alpha$ coincides with the original tuple $(\mathfrak{X}/\mathcal{O}_K, \{U_i\}, f_i)$.

When K is of mixed characteristic $(0, p)$, the situation is slightly different but a similar construction exists. By Lemma 3.3, we may assume that \mathcal{O}_K is complete and the residue field F is perfect (see also Remark 3.4). Then, by [SGA7-I], I, (0.5.3), there exists a complete discrete valuation ring R contained in \mathcal{O}_K such that $\pi \in R$, the residue field of R is a purely inseparable extension of a finitely

generated extension of \mathbb{F}_p , and we can write \mathcal{O}_K as a filtered inductive limit $\mathcal{O}_K = \varinjlim A_\alpha$, where each A_α is a finitely generated smooth R -algebra. Then, there exists a tuple $(\mathfrak{Y}/A_\alpha, \{V_i\}, g_i)$ with the same properties as above such that the pullback of it by $\text{Spec } \mathcal{O}_K \rightarrow \text{Spec } A_\alpha$ coincides with the original tuple $(\mathfrak{X}/\mathcal{O}_K, \{U_i\}, f_i)$.

In any case, $(\pi = 0)$ is a regular divisor of $\text{Spec } A_\alpha$. Let $s \in \text{Spec } A_\alpha$ be the image of the closed point of $\text{Spec } \mathcal{O}_K$, and $(A_\alpha)_s$ the localization of A_α at s . There exist elements $a_1, \dots, a_r \in A_\alpha$ such that $\{\pi, a_1, \dots, a_r\}$ is a regular system of parameters of the local ring $(A_\alpha)_s$. Let B be the henselization of the quotient $(A_\alpha)_s/(a_1, \dots, a_r)$. Then, B is a henselian discrete valuation ring and the image of π in B is a uniformizer. We put $\mathfrak{Z} := \mathfrak{Y} \otimes_{A_\alpha} B$.

In conclusion, we have the following cartesian diagram:

$$(4.1) \quad \begin{array}{ccccc} \mathfrak{X} & \longrightarrow & \mathfrak{Y} & \longleftarrow & \mathfrak{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } \mathcal{O}_K & \longrightarrow & \text{Spec } A_\alpha & \longleftarrow & \text{Spec } B, \end{array}$$

such that \mathcal{O}_K, B are henselian discrete valuation rings, and the images of the closed points of $\text{Spec } \mathcal{O}_K, \text{Spec } B$ coincide with $s \in \text{Spec } A_\alpha$. Hence the geometric special fibers of $\mathfrak{X}, \mathfrak{Z}$ are the same. The scheme \mathfrak{Y} is proper over $\text{Spec } A_\alpha$ and Zariski locally étale over $\text{Spec } A_\alpha[X_0, \dots, X_n]/(X_0 \cdots X_r - \pi)$ for some r . Hence \mathfrak{Z} is a proper strictly semistable scheme over B . Finally, when K is of equal characteristic $p > 0$ (resp. equal characteristic 0), the residue field of B is finitely generated over \mathbb{F}_p (resp. \mathbb{Q}). When K is of mixed characteristic $(0, p)$, the residue field of B is a purely inseparable extension of a finitely generated extension of \mathbb{F}_p .

5. DEGENERATION OF THE WEIGHT SPECTRAL SEQUENCES

In this section, we prove that the weight spectral sequences degenerate at E_2 without any assumption on residue fields. This result was already obtained by C. Nakayama by using log geometry ([Na], Theorem 0.1). Here we give another proof without using log geometry. We prove it by comparing the weight spectral sequences of $\mathfrak{X}/\mathcal{O}_K$ and \mathfrak{Z}/B in the diagram (4.1) in §4.

Proposition 5.1 ([Na], Theorem 0.1). *Let notation be as in §1. Then, the weight spectral sequence of $\mathfrak{X}/\mathcal{O}_K$ degenerates at E_2 .*

Proof. Let us consider the diagram (4.1) in §4. By Proposition 2.1, the weight spectral sequence for \mathfrak{Z}/B degenerates at E_2 . Since $\mathfrak{X}/\mathcal{O}_K$ and \mathfrak{Z}/B have the same geometric special fiber, the left hand sides of the equality in Lemma 3.1 for $\mathfrak{X}/\mathcal{O}_K, \mathfrak{Z}/B$ are the same. Hence, by Lemma 3.1, it is enough to show that the right hand sides for $\mathfrak{X}/\mathcal{O}_K, \mathfrak{Z}/B$ are also the same. By [SGA4-III], XVI, Corollaire 2.2, this follows from the fact that \mathfrak{Y} is proper and smooth outside

$(\pi = 0)$ and the images of the generic points of $\text{Spec } \mathcal{O}_K, \text{Spec } B$ do not lie on the divisor $(\pi = 0)$. \square

6. PROOF OF THEOREM 1.1

Here we shall prove Theorem 1.1. The first part of Theorem 1.1 is already proved in Proposition 5.1.

Let notation be as in §1. Assume that K is of equal characteristic, and consider the diagram (4.1) in §4. According to Lemma 3.2, it is enough to prove Conjecture 1.2 for \mathfrak{X}/B . Hence, we may replace $\mathfrak{X}/\mathcal{O}_K$ by \mathfrak{X}/B .

Recall that, if K is of characteristic $p > 0$ (resp. characteristic 0), A_α in the diagram (4.1) is a finitely generated smooth $\mathbb{F}_p[\pi]_{(\pi)}$ -algebra (resp. $\mathbb{Q}[\pi]_{(\pi)}$ -algebra), and \mathcal{O}_K is the henselization of the discrete valuation ring $(A_\alpha)_s/(a_1, \dots, a_r)$. Therefore, there exist a finitely generated smooth $\mathbb{F}_p[\pi]$ -algebra (resp. $\mathbb{Z}[\pi, 1/l]$ -algebra) A contained in \mathcal{O}_K and a scheme $f: \tilde{\mathfrak{X}} \rightarrow \text{Spec } A$ such that \mathcal{O}_K is the henselization of A at (π) , $(\pi = 0)$ is a regular divisor of $\text{Spec } A$, f is proper and smooth outside $(\pi = 0)$, and $\tilde{\mathfrak{X}} \otimes_A \mathcal{O}_K \cong \mathfrak{X}$.

To prove Theorem 1.1, it is enough to prove the following proposition.

Proposition 6.1. *Let $\mathfrak{X}/\mathcal{O}_K$ and $f: \tilde{\mathfrak{X}} \rightarrow \text{Spec } A$ be as above. Then, Conjecture 1.2 holds for $\mathfrak{X}/\mathcal{O}_K$.*

Proof. Let M be the monodromy filtration on $V := H_{\text{ét}}^w(\mathfrak{X}_{K^{\text{sep}}}, \mathbb{Q}_l)$ as in §2.1. We can construct an l -adic sheaf version of (V, M) as follows (for details, see [De3], Variante 1.7.8). Since the action of $\text{Gal}(K^{\text{sep}}/K)$ on V is tamely ramified ([RZ]), V extends to a smooth l -adic sheaf on $\varprojlim \text{Spec } \mathcal{O}_K[\pi^{1/l^m}]$ with $\mathbb{Z}_l(1)$ -action. Therefore, the smooth l -adic sheaf $R^w f_* \mathbb{Q}_l$ on $(\text{Spec } A) \setminus (\pi = 0)$ extends to a smooth l -adic sheaf on $\varprojlim \text{Spec } A[\pi^{1/l^m}]$ with $\mathbb{Z}_l(1)$ -action by Zariski-Nagata's purity ([SGA2], X, Théorème 3.4). There is a natural section of $\varprojlim \text{Spec } A[\pi^{1/l^m}] \rightarrow \text{Spec } A$ on $(\pi = 0)$ because $\text{Spec } A[\pi^{1/l^m}] \rightarrow \text{Spec } A$ is totally ramified along $(\pi = 0)$. By pulling back, we have an l -adic sheaf \mathcal{F} on $(\pi = 0)$ with $\mathbb{Z}_l(1)$ -action. By the same way as in §2.1, we have a filtration \mathcal{M} on \mathcal{F} such that the pair $(\mathcal{F}, \mathcal{M})$ specializes to (V, M) in an obvious way.

Since A is a smooth $\mathbb{F}_p[\pi]$ -algebra (resp. $\mathbb{Z}[\pi, 1/l]$ -algebra), for a closed point $t \in (\pi = 0) \subset \text{Spec } A$, there exists a curve $C \subset \text{Spec } A$ over a finite field which passes through t and intersects transversally with $(\pi = 0)$. Then, the filtration \mathcal{M} on \mathcal{F} specializes to the monodromy filtration on the restriction of $R^w f_* \mathbb{Q}_l$ to $C \setminus (\pi = 0)$ at $t \in C$ in the sense of [De3], (1.7.2). Since C is a curve over a finite field, the stalk $(\text{Gr}_k^{\mathcal{M}} \mathcal{F})_t$ has weight $w + k$, and the smooth l -adic sheaf $\text{Gr}_k^{\mathcal{M}} \mathcal{F}$ on $(\pi = 0)$ has weight $w + k$ (Remark 2.4, see also [De3], Théorème 1.8.4, Corollaire 1.8.7). Since F is the function field of the divisor $(\pi = 0)$ and $\text{Gr}_k^{\mathcal{M}} V$ is the stalk of $\text{Gr}_k^{\mathcal{M}} \mathcal{F}$ at the generic point of $(\pi = 0)$, Conjecture 2.3 holds for \mathfrak{X}_K (see also

our definition of the notion of weights in §2.2). Therefore, by Proposition 2.5, Conjecture 1.2 holds for \mathfrak{X} .

Hence the assertion of Proposition 6.1 is proved and the proof of Theorem 1.1 is complete. \square

Remark 6.2. The above proof is valid only in the equal characteristic case. By the same argument as above, we can prove that Conjecture 2.3 in mixed characteristic $(0, p)$ implies Conjecture 1.2 in mixed characteristic $(0, p)$. However, we can not prove Conjecture 1.2 in mixed characteristic. The problem is as follows. In mixed characteristic $(0, p)$, A is a finitely generated smooth R -algebra, where R is a complete discrete valuation ring of mixed characteristic $(0, p)$ as in §4. Since R is of mixed characteristic $(0, p)$, all characteristic $p > 0$ points of $\text{Spec } A$ are lying over the closed point of $\text{Spec } R$. Therefore, all curves $C \subset \text{Spec } A$ over a finite field are contained in the divisor $(\pi = 0)$, and there is no curve $C \subset \text{Spec } A$ over a finite field intersecting transversally with $(\pi = 0)$.

7. SOME CONSEQUENCES

7.1. The equal characteristic case of Conjecture 2.3. By a similar specialization argument, we can also prove the equal characteristic case of Conjecture 2.3 (for the finite residue field case, see also [Te]).

Proposition 7.1. *Let notation be as in the beginning of §2. If K is of equal characteristic and F is a purely inseparable extension of a finitely generated extension of a prime field, then Conjecture 2.3 holds.*

Proof. By Grothendieck's monodromy theorem, after replacing K by a finite extension of it, the action of $\text{Gal}(K^{\text{sep}}/K)$ on $V := H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ is tamely ramified and the action of the inertia group I_K on V is unipotent (see §2.1). By the same way as in §4, if K is of characteristic $p > 0$ (resp. characteristic 0), there exist a finitely generated smooth $\mathbb{F}_p[\pi]$ -algebra (resp. $\mathbb{Z}[\pi, 1/l]$ -algebra) A contained in \mathcal{O}_K and a scheme $f: \tilde{\mathfrak{X}} \rightarrow \text{Spec } A$ such that f is proper and smooth outside $(\pi = 0)$ and $\tilde{\mathfrak{X}} \otimes_A K \cong X_K$. Here we do not make any assumption on the fiber of f over $(\pi = 0)$. Then, by the same way as in the proof of Proposition 6.1, there exist a smooth l -adic sheaf \mathcal{F} on $(\pi = 0)$ and a filtration \mathcal{M} on it such that the pullback of $(\mathcal{F}, \mathcal{M})$ by $\text{Spec } \mathcal{O}_K \rightarrow \text{Spec } A$ is (V, M) , and $\text{Gr}_k^{\mathcal{M}} \mathcal{F}$ has weight $w+k$ in the usual sense. Since F is an extension of the residue field $\kappa(s)$ at some point $s \in (\pi = 0)$, $\text{Gr}_k^{\mathcal{M}} V$ has weight $w+k$ in the sense of §2.2. Hence Conjecture 2.3 holds for X_K . \square

Remark 7.2. It is also possible to deduce Proposition 7.1 from Theorem 1.1 as follows. By de Jong's alteration ([dJ], Theorem 6.5), after replacing K by a finite extension of it, there is a proper strictly semistable scheme \mathfrak{Y} over \mathcal{O}_K , and a proper surjective generically finite morphism $g: \mathfrak{Y}_K \rightarrow X_K$. We put $V := H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ and $V' := H_{\text{ét}}^w(\mathfrak{Y}_{K^{\text{sep}}}, \mathbb{Q}_l)$. Since \mathfrak{Y}_K, X_K are proper and smooth

over K , there is a $\text{Gal}(K^{\text{sep}}/K)$ -equivariant trace map $g_*: V' \rightarrow V$. We know that the composite $g_* \circ g^*$ is the multiplication-by- d map on V , where d denotes the separable degree of g at the generic point of X_K . Hence $g^*: V \rightarrow V'$ is injective and V is a direct summand of V' as $\text{Gal}(K^{\text{sep}}/K)$ -representations. Therefore, the restriction of the monodromy (resp. weight) filtration on V' to V is the monodromy (resp. weight) filtration on V . By Theorem 1.1 and Proposition 2.5, Conjecture 2.3 holds for \mathfrak{Y}_K . Hence Conjecture 2.3 holds also for X_K .

7.2. A Hodge analogue over \mathbb{C} . When $F = \mathbb{C}$, Steenbrink originally considered a Hodge analogue of Conjecture 1.2 ([St]). As an application of Theorem 1.1, we can give a modulo $p > 0$ reduction proof of it.

Proposition 7.3 ([St], Proposition 5.14). *Let notation be as in [St], §5. Then, Schmid's limit Hodge structures coincide with Steenbrink's limit Hodge structures.*

Proof. It is enough to prove [St], Theorem 5.9 which is nothing but a Hodge analogue of Conjecture 1.2. Hence, by the comparison theorem between étale and singular cohomology ([SGA4-III], XI, Théorème 4.4), this follows from Theorem 1.1. \square

Remark 7.4. Note that, Steenbrink's original argument in [St], Theorem 5.9 is incomplete (see [SaM], 4.2.5, [SaZ], §2.3, [GNA]).

Remark 7.5. Conversely, by Lefschetz principle, if K is of equal characteristic 0, Conjecture 1.2 follows from the results over \mathbb{C} . Therefore, we have two completely different proofs of Conjecture 1.2 in the equal characteristic 0 case. Illusie pointed out to the author that there is a technical difference between them. In the Hodge theoretic proof, the proper case is reduced to the projective case by Chow's lemma because polarized Hodge structures are used (see [SaZ], Remarks (0.5), (iv)). On the other hand, in this paper, we can directly treat the proper case because Deligne's results in [De3] are valid without the projectivity assumption.

REFERENCES

- [Ar] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. No. 36, (1969), 23–58.
- [De1] P. Deligne, *Théorie de Hodge I*, in *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 1, 425–430, Gauthier-Villars, Paris, 1971.
- [De2] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. No. 43, (1974), 273–307.
- [De3] P. Deligne, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. No. 52, (1980), 137–252.
- [GH] U. Görtz, T. Haines, *The Jordan-Hölder series for nearby cycles on some Shimura varieties and affine flag varieties*, preprint, math.AG/0402143, 2004.
- [GNA] F. Guillén, V. Navarro Aznar, *Sur le théorème local des cycles invariants* Duke Math. J. **61** (1990), no. 1, 133–155.
- [Il1] L. Illusie, *Autour du théorème de monodromie locale*, Périodes p -adiques (Bures-sur-Yvette, 1988), Astérisque No. 223, (1994), 9–57.

- [Il2] L. Illusie, *An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology*, in *Cohomologies p -adiques et applications arithmétiques, II*, Astérisque No. 279 (2002), 271–322.
- [It1] T. Ito, *Weight-monodromy conjecture over positive characteristic local fields*, Master's thesis, University of Tokyo, 2001.
- [It2] T. Ito, *Weight-monodromy conjecture for certain threefolds in mixed characteristic*, Internat. Math. Res. Notices **2004**, no. 2, 69–87.
- [It3] T. Ito, *Weight-monodromy conjecture for p -adically uniformized varieties*, math.NT/0301201, 2003, to appear in Invent. Math.
- [dJ] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. No. 83, (1996), 51–93.
- [Na] C. Nakayama, *Degeneration of l -adic weight spectral sequences*, Amer. J. Math. **122** (2000), no. 4, 721–733.
- [Ra] M. Rapoport, *On the bad reduction of Shimura varieties*, in *Automorphic forms, Shimura varieties, and L -functions, Vol. II (Ann Arbor, MI, 1988)*, 253–321, Academic Press, Boston, MA, 1990.
- [RZ] M. Rapoport, T. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), no. 1, 21–101.
- [SaM] M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989).
- [SaZ] M. Saito, S. Zucker, *The kernel spectral sequence of vanishing cycles*, Duke Math. J. **61** (1990), no. 2, 329–339.
- [SaT] T. Saito, *Weight spectral sequences and independence of ℓ* , J. Inst. Math. Jussieu **2** (2003), 583–634.
- [Sc] W. Schmid, *Variation of Hodge structure : the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.
- [Se] J.-P. Serre, *Corps locaux*, Deuxieme édition, Hermann, Paris, 1968.
- [ST] J.-P. Serre, J. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.
- [St] J. Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1975/76), no. 3, 229–257.
- [Te] T. Terasoma, *Monodromy weight filtration is independent of l* , preprint, math.AG/9802051, 1998.
- [EGA3] *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.*, Inst. Hautes Études Sci. Publ. Math. No. 11 (1961), 167 pp.
- [SGA2] *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, Adv. Stud. Pure Math., 2, North-Holland, Amsterdam, 1968.
- [SGA4-III] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Math., 305, Springer, Berlin, 1973.
- [SGA7-I] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Math., 288, Springer, Berlin, 1972.

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